International Journal of Engineering Technology and Scientific Innovation

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

NUMERICAL SCHEME BASED ON INTERPOLATION FUNCTION FOR SOLVING STIFF DIFFERENTIAL EQUATIONS

A. FOSU

Department of Pure & Applied Mathematics; Faculty of Science, Engineering and Technology; Walter Sisulu University; Private Bag X1, Mthatha 5117; Republic of South Africa.

ABSTRACT

The initial value problems with stiff ordinary differential equation systems occur in many fields of engineering, particularly in the studies of electrical circuits, vibrations, chemical reactions and also in many non-industrial areas like weather prediction.

The aim of this study is to propose a one-step numerical scheme that can solve some of these problem of stiff ordinary differential equations. The derivation of the scheme is based on interpolating functions.

The efficiency of the method is examined in terms of consistency, stability and convergence as well as construct the Region of Absolute Stability (RAS) of the scheme.

Keywords: Consistence, convergence, region of absolute stability, stiff ordinary differential equations.

1. INTRODUCTION

Various fields of engineering and science include a special class of differential equations taken up in the initial valve problems termed as stiff differential equations result from the phenomena with widely differing time scales. A set of differential equations is ”stiff” when an excessively small step is needed to obtain correct integration. With stiff differential equations occuring in many fields such as engineering, biology, weather prediction, it is required to solve them efficiently. Almost all stiff differential equations can not be solved analytically, therefore requiring numerical procedure. Traditional methods such as Euler, explicit Runge-Kutta and others are restricted to a very small step size in order for the solution to be stable. This means that a great deal of computer time could be required.

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

Although over the years a number of methods have been developed for solving stiff ordinary differential equations [[4],[9], [21],[22]] just to mention a few, most of these methods concentrated on the rigorous derivation of the method and not much on applying to solve typical problems. A new numerical scheme is proposed for solving some of these stiff ordinary differential equations. The method developed is applied to typical problems. The rest of the article is organized as follows: Section 2 deals with the development of the numerical scheme while consistency, convergence and stability of the scheme are developed in Section 3. In Section 4 numerical experiments on some test problems on the performance of the scheme with the exact solutions are discussed.

2. DEVELOPMENT OF THE NEW NUMERICAL METHODS

A numerical method for solving stiff initial value problem based on non-linear interpolating function was developed in this section.

2.1. The Interpolating function. The mesh points are defined as interval $[a,b]$ in the usual way, $x_n = a + nh, n = 0,1,2,...$, and let y_n represent the numerical estimate to the theoretical value $y(x_n)$ and f_n represent $f(x_n,y_n)$. Let us assume that the theoretical solution $y(x)$ to the stiff initial value problem can be locally represented in the interval $[x_n,x_{n+1}]$ by the interpolating function

(2.1) $$F(x) = a + bx + cx^2 + de^{x3},$$

where a, b, c, and d are constants.

2.2. The Imposed Constraints. We shall make the following assumptions:-

(i) that the interpolating function coincides with the theoretical solution at $x = x_n$ and $x = x_{n+1}$, i.e.,

(2.2) $$F(x_n) = a + bx_n + cx^2_n + de^{x3n},$$

and

(2.3) $$F(x_{n+1}) = a + bx_{n+1} + cx^2_{n+1} + de^{x3n+1},$$

(ii) that the first, second, third and fourth derivatives with respect to x of the interpolating function coincide respectively with the differential equation as well as its first and second derivatives with respect to x at $x = x_n$, i.e.,

(2.4) $$F'(x_n) = f_n$$

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

(2.5) $F''(xn) = fn'$

(2.6) $F'''(xn) = fn''$.

This implies that

$x3n$ 2

(2.7) $b + 2cx_n + 3de \quad x = f_n$

(2.8) $2c + d(6e^{x_n^3}x_n + 9e^{x_n^3}x_n^4) = f_n'$

(2.9) $d(6e^{x_n^3} + 54e^{x_n^3}x_n^3 + 27e^{x_n^3}x_n^6) = f_n''$.

The system of equations in (2.7) to (2.9) is solved to obtain values of b, c, and d. We obtain

(2.10) $$c = \frac{(2 + 18x_n^3 + 9x_n^6)f_n' - x_n(2 + 3x_n^3)f_n''}{4 + 36x_n^3 + 18x_n^6}$$

(2.11) $$d = \frac{e^{-x_n^3}f_n''}{6 + 54x_n^3 + 27x_n^6}$$

$b = [(2 + 18x^3{}_n + 9x^6{}_n)f_n + x - n(-(2 + 18x^3{}_n + 9x_n{}^6)f_n{}' + x_n(1 + 3x_n{}^3)f_n{}'']$
[

−1

×3 $_6$]

2 $+ 18x_n + 9x_n$

(2.12)

2.3. The Numerical Scheme. If we subtract equation (2.2) from (2.3), we obtain

(2.13) $y_{n+1} - y_n = F(x_{n+1}) - F(x_n)$

(2.14) $y_{n+1} - y_n = b(x_{n+1} - x_n) + c(x_{n+1}^2 - x_n^2) + d(e^{x^3_{n+1}} - e^{x_n^3})$.

If we now put $x_n = x_0 + nh$ and $x_{n+1} = x_0 + (n+1)h$, we can write our one-step method as

(2.15) $yn+1 - yn = b(h) + c(2x0h + h2 + 2hnh) + d(e(x0+nh+h)3 - e(x0+nh)3)$.

Substituting for b, c, and d in (2.15), we obtain

$y_{n+1} - y_n$ = [6h(2 + 18h^3n^3 + 9h^6n^6)f_n + 3h^2(2 + 18h^3n^3 + 9h^6n^6)f_{n1}
− h^3(1+3n+3n^2) − 6 4 − 3]

+(2 + 2e 9h n 6h n(1 + n))f_{n2} 3 3 6 6]
−1 [

(2.16) ×6(2 + 18h n + 9h n)

Equation (2.16) is the required one-step method. A general one-step method is given in the form

(2.17) $y_{n+1} = y_n + h\phi(x_n,y_n;h)$,

where $\phi(x_n,y_n;h)$ is called the incremental function of the method. Analysis of the incremental function is carried out to determine if the scheme is convergent and consistent. We now derive the incremental function of our scheme. A slight rearrangement of (2.16) and application of Taylor series expansion leads to

(2.18) $$y_{n+1} = y_n + f_n h + \frac{f'_n h^2}{2} + \frac{f''_n h^3}{6} + \frac{f'''_n h^4}{24} + O(h^5).$$

Ignoring higher order terms in (2.18) and comparing the expression with (2.17) we obtain that

(2.19) $$\phi(x_n, y_n; h) = f_n + \frac{f'_n h}{2} + \frac{f''_n h^2}{6} + \frac{f'''_n h^3}{24}.$$

3. ANALYSIS OF THE NEW SCHEME

Every developed numerical scheme is supposed to meet certain standard requirements, and should "compare well" with other known methods([11], [6],[7]). The central concepts commonly used in the analysis are: *Convergence*, *Consistency*, *Order* and *Stability*.

3.1. Consistency and Order. Consistency and Order is easily shown via a principle of (**Fatunla**([19]), A numerical scheme with an incremental function $\phi(x_n,y_n;h)$ is said to be consistent with the initial value problem if

(3.1) $$\phi(x_n,y_n;0) = f(x_n,y_n).$$

This is easily shown to be the case for the incremental function (2.19).

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

3.2. Convergence and Stability. Similarly, convergence and stability of the constructed numerical scheme is easily established (**Lambert**([12]))

3.3. Region of absolute stability (RAS). To obtain the region of absolute stability for the developed scheme, we apply the scheme (2.16) to the test problem (2.18). We proceed to determine f_n, f_n', f''^n and substitute in equation (2.16). Note that for (2.18) $f(x,y) = \lambda y$, so that at the points $x = x_n$, we have $f_n = \lambda y_n$. By using the Mathematica package ([16]) we obtain the following:

(3.2) $$f(x_n) = \lambda y_n$$

(3.3) $$f'(x_n) = \lambda^2 y_n$$

(3.4) $$f''(x_n) = \lambda^3 y_n$$

(3.5) $$f'''(x_n) = \lambda^4 y_n$$

(3.6) $$y_{n+1} = y_n + f_n h + \frac{f_n' h^2}{2} + \frac{f_n'' h^3}{6} + \frac{f_n''' h^4}{24} = R(\lambda h) y_n,$$

where

(3.7) $$R(\lambda h) y_n = \left(\frac{24 + 24h\lambda + 12h^2\lambda^2 + 4h^3\lambda^3 + h^4\lambda^4}{24} \right) y_n$$

with

(3.8) $$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24},$$

which is the stability polynomial of the method. The region of absolute stability is the region (in the complex plane) defined by $| 1 + R(z) | \leq 1$ (for a rigorous definition of stability, see ([5]). Figure (1) shows the region of absolute stability for the scheme (2.16) obtained by using the Mathematica package ([16]).

4. COMPUTER EXPERIMENTS AND RESULTS

In this section the constructed numerical method,

(4.1) $$y_{n+1} = y_n + f_n h + \frac{f_n' h^2}{2} + \frac{f_n'' h^3}{6} + \frac{f_n''' h^4}{24} + 0(h^5).,$$

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

is applied to a number of stiff initial value problems to illustrate the performance of the method. We benchmark its performance against the exact solution (where we have one).

The implementation of the method is done using MATLAB ([2]) and Mathematica 5.0.

For Examples 1 to 3 that are presented below, the solution found using (4.1) is compared with the exact solutions. Results are tabulated and corresponding graphs of solutions are given side by side.

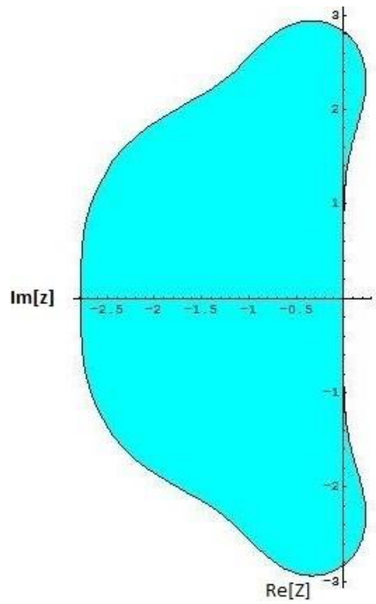

Figure 1: Region of absolute stability for the numerical scheme

4.1. Example 1: $y' = -20(y - t^2) + 2t$, $y(0) = \frac{1}{3}$. The exact solution for the problem is

$$y(x) = t^2 + \frac{1}{3}e^{-20t}.$$

To find a numerical solution using (2.17), we firstly determine f', and f''. We obtain

$$\begin{aligned} f(x,y) &= 2(x + 10x^2 - 10y) \\ f'(x,y) &= 2 + 400(-x^2 + y) \\ f''(x,y) &= 800(x^2 - y) \end{aligned} \quad (4.2)$$

The results are presented in Table 1 and Figure 2, comparing solutions from (4.1) - column "new scheme"; and the exact solution - column "Exact soln",

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

t	New scheme	Exact soln
0.90	0.809983	0.810000
1.00	1.000006	1.000000
1.10	1.209998	1.210000

Table 1: Table of results for Example 1.

4.2. Example 2: $y' = -15(y - t^{-3})$, $y(1) = 0$. The exact solution to the problem is

$$y(x) = -e{-}15t + e{-}3.$$

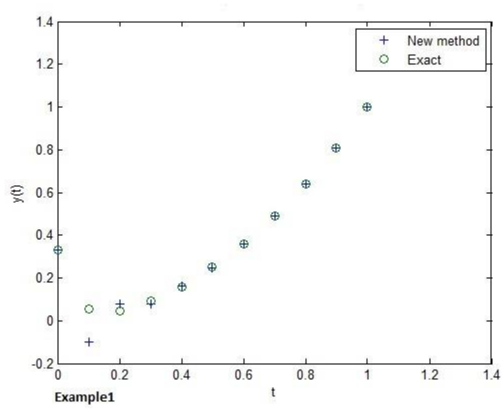

Figure 2: Graph of results in Table 1

Proceeding as we did in Example 1 above, we have that

$$
\begin{aligned}
f(x,y) &= \frac{(-3(1-5x+x^4y))}{x^4} \\
f'(x,y) &= \frac{12}{x^5} + 225(-x^{-3}+y) \\
f''(x,y) &= \frac{-60}{x^6} - 3375(-x^{-3}+y)
\end{aligned}
\quad (4.3)
$$

The results are presented in Table Table 2 and Figure 3.

t	New scheme	Exact soln
1.90	0.145769	0.145794
2.00	0.124983	0.125000
2.10	0.107968	0.107980

Table 2: Table of results for Example 2.

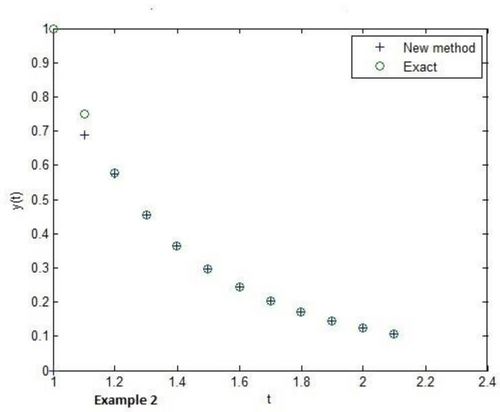

Figure 3: Graph of results in Table 2

4.3. Example 3: $y' = -100(y - x^3 + 3x^2)$, $y(0) = 0$. The exact solution is

$$y(x) = x^3.$$

We have

$$\begin{aligned} f(x,y) &= 3x^2 + 100x^3 - 100y \\ f'(x,y) &= 6x + 10000(-x^3 + y) \\ f''(x,y) &= 6 + 1000000(x^3 - y) \end{aligned} \tag{4.4}$$

The results for the solution of this stiff IVP are presented in Table 3 and Figure 4.

t	New scheme	Exact soln
0.09	0.000729	0.000729
0.10	0.001000	0.001000
0.11	0.001331	0.001331

Table 3: Table of results for Example 3.

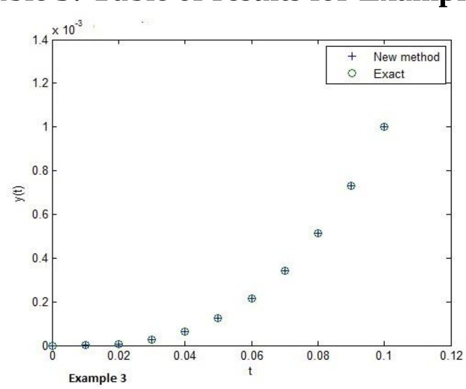

Figure 4: Graph of results in Table 3

ISSN: 2456-1851

Volume: 04, Issue: 05 "September-October 2019"

5. CONCLUDING REMARKS

In this study a numerical method have been developed for solving some stiff differential equations that arises from physical and engineering fields. In particular the consistency, the convergence was checked and determined the region of absolute stability of the method. Furthermore, the performance of the method was illustrated by applying it to three typical examples. The results of the method with that of the exact solution was illustrated in tables with their graphs.

One could experiment with different types of functions to see what other possible numerical schemes could be derived.

REFERENCES

A.FOSU and E. A.IBIJOLA, *On the development and performance of interpolation function based methods for numerical solution of ordinary differential equations.* International Journal of Numerical Methods and Applications, 15(3),(2016), pp 197-227.

A. GILAT,*MATLAB: An Introduction with Applications,* (2004),John Wiley & Sons, Michigan.

A. S. ACKLEH, R.B. KEARFOTT and E.J. ALLEN, *Classical and Modern Numerical Analysis: Theory, Methods and Practice*, Chapman & Hall/CRC, Boca Raton, (2003).

B.L. EHLE, *On Pade approximations to the exponential function and A-stable methods for the numerical solution of initial value problems.* SIAM J. Math. Anal., 4(1973),pp 671-680.

C.W. GEAR, *Numerical Initial-Value Problems in Ordinary Differential Equations*, (1971), Prentice-Hall, Englewood Cliffs, N.J.

E.A. IBIJOLA and P. KAMA, *On the convergence, consistency and stability of a one-step method for numerical integration of ordinary differential equation*, International Journal of Computer Mathematics, **73**(2), (1999), pp 261–277.

E.A. IBIJOLA and J. SUNDAY, *A comparative study of standard and exact finite difference schemes for numerical solution of ordinary differential equations*, Australian Journal of Basic and Applied Sciences, **4**(4), (2010),pp 624–632.

F.SCHIED, *Theory and Problems of Numerical Analysis*, McGraw-Hill, New York, (1968).

J.C. BUTCHER, *Implicit Runge-Kutta processes,* Math. Comp. 18, (1964), pp 50-64.

J.C. BUTCHER, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, (2003).

J.D. FAIRES and R.L. BURDEN, *Numerical Methods*, PWS Publishing Company, Boston, (1993).

J.D. LAMBERT, *Computational Methods in Ordinary Differential Equations*, Wiley, New York, (1973).

N.P. BALI, *Differential Equations*, Firewall Media, NewDelhi, (2010).

P. HENRICI, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York, (1962).

S. THOHURA and A. RAHMAN, *Numerical Approach for Solving Stiff Differential Equations: A Comparative Study*, Global Journals Inc. **13**(6), USA, (2013).

S. WOLFRAM, *Mathematica: A System for Doing Mathematics by Computer*, Addison Wesley, New York, (1991).

S.D. CONTE and C. DE BOOR, *Elementary Numerical Analysis: An Algorithmic Approach*, McGraw-Hill, New York, (1980).

S.O. FATUNLA,*A new Algorithm for numerical solution of Ordinary Differential Equations*, Computer and Mathematics with Applications. Pergamon Press,2, (1976),PP 247-253.

S.O. FATUNLA, *Numerical Methods for Initial Value Problems in Ordinary Differential Equations*, Academic Press, New York, (1988).

W. CHENEY and D. KINCAID, *Numerical Mathematics and Computing*, Brooks/Cole, New York, (1999).

W.H. ENRIGHT, T.E. HULL and B. LINDBERG, (1975) *Comparing numerical methods for stiff systems of ODEs.* BIT, 15, (1975), pp 10-48.

W.L. MIRANKER, *Numerical Methods for Stiff Equations and Singular Perturbation Problems.* D. Reidel, Dordrecht, Holland, (1981).

